

Roles of Interfering Radiation Emitted from Decaying Pulses Obeying Soliton Equations Belonging to the Ablowitz-Kaup-Newell-Segur Systems

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The nonlinear Schrödinger (NLS) equation under the box-type initial condition, which models general multiple pulses deviating from pure solitons, is analyzed. Following the approximation by splitting the initial pulse into many small bins [G. Boffetta and A. R. Osborne, *J. Comp. Phys.* **102**, 25 (1992)], we can analyze the Zakharov-Shabat eigenvalue problem to construct transfer matrices connecting the Jost functions in each interval without direct numerical computation. We can obtain analytical expressions for the scattering data that describe interfering radiation emitted from initial pulses. The number of solitons that appear in the final stage is predicted theoretically, and the condition generating an unusual wave such as a double-pole soliton is derived. Numerical analyses under box-type initial conditions are also performed to show that the interplay between the tails from decaying pulses can affect the asymptotic profile.

1. Introduction

The theory of solitons has played a prominent role in the development of mathematical physics.^{1–6)} It has been applied to many interesting fields of physics, ranging from high-energy and gravitational physics^{7–12)} to the physics of more experimentally accessible energy scales such as fluid mechanics and plasma physics.^{13–18)} In the context of condensed matter or low-temperature physics, Bose-Einstein condensed (BEC) systems are of particular interest and have attracted considerable attention. In BEC systems, the macroscopic wave functions of the condensates are known to obey the nonlinear Schrödinger equation (NLSE) of the

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third order, whose nonlinear term represents the biparticle collision of constituent atoms.^{19–22)} The absolute square of the macroscopic wave function is interpreted as the particle density of the constituent atoms and is directly observable by an optical method. Furthermore, by developing many techniques to control the system, experimentalists have already realized various geometries by applying ingenious external potentials to the BEC systems. Quasi-one-dimensional potentials are among them, and self-focusing bright solitons have been realized by a number of laboratories.^{23–28)} Recently, collisions between such BEC solitons have been experimentally examined and reported.^{29–31)} Similar experiments are also promising possible in the field of nonlinear optics.^{32–36)}

In real experiments, however, one cannot generate pure solitons that are the exact solutions of the NLSE, and the word “solitons” is often used to mean multiple pulses of condensates or photons. Realistically, one expects that the multiple pulses that deviate from pure solitons decay, emitting radiation, and fall into a final state that includes pure solitons after some transition time. The investigation of the roles played by the interaction between the emitted radiation is the motivation of this study. For example, even if each constituent pulse is too small to sustain a soliton, the interfering radiation and some nonlinear effects can generate a large amplitude. It is expected to observe the creation of new solitons.

The time evolution of such nonlinear systems is investigated by solving the initial value problems of corresponding soliton equations. The inverse scattering transform (IST) is a useful method that can deal with such problems.^{1–6)} This method is based on a scattering problem of a set of auxiliary linear equations, which are associated with the original soliton equation:

$$\Psi_x = M\Psi, \quad (1a)$$

$$\Psi_t = N\Psi, \quad (1b)$$

where the quantities M and N are matrices or operators including the unknown functions of the soliton equation and the spectral parameter. The wave function Ψ represents an auxiliary field obeying appropriate boundary conditions. An important step of the IST method is to analyze the spatial equation (1a) as a scattering problem, whose potential term is given by the initial condition of the unknown function. The wave function Ψ and the spectral parameter correspond to the eigenfunction and eigenvalue, respectively. This is called the Zakharov-Shabat (ZS) problem.³⁷⁾ Once the ZS problem is solved, the time-evolved wave function is easily obtained through Eq. (1b), and the solution of the Cauchy problem is provided by virtue of the Gel'fand-Levitan-Malchenko (GLM) equation.³⁸⁾ The GLM equation clearly shows that the number of discrete eigenvalues determines that of solitons to be generated

asymptotically.

Usually, the ZS problems accompanied by general initial conditions are difficult to solve, and it is rarely possible for us to predict how many solitons remain in the final state, except under pure soliton initial conditions. Concerning this problem, in 1991, Boffetta and Osborne proposed³⁹⁾ an approximation method to obtain the scattering data for arbitrary initial wave packets by discretizing the spatial coordinate. Obviously, this method is applicable to other Ablowitz-Kaup-Newell-Segur (AKNS) soliton equations.⁴⁰⁾ In this paper, we consider a set of box-type potentials as models for multiple pulses that are not pure solitons by applying this method. We can obtain an analytical expression that describes interfering radiation emitted from decaying original pulses. In addition, we can extract information such as the number of solitons that appear in the final state. We find that the interplay between the diffusing tails from decaying pulses can affect the asymptotic profile markedly, and this is confirmed by directly integrating the NLSE by numerical simulation. Furthermore, we derive the parameter conditions that generate double-pole solitons.⁴¹⁾

This paper is organized as follows. In the next section, we briefly summarize the IST method and the ZS problem taking the NLS equation as an example. In Sect. 3, we introduce Boffetta and Osborne's method and explain how to extract approximated scattering data. In Sect. 4, distributions of eigenvalues for the NLSE with double-box-type initial conditions are derived. We also show that this simple application leads to some nontrivial results including conditions for generating double-pole solitons and the crucial roles played by interfering radiation from each boxlike pulse. The results of our numerical simulation are shown in Sect. 5. The final section is devoted to discussion and concluding remarks.

2. Summary of the IST Method and ZS Problem

We give a brief explanation of the IST method and ZS problem for later use. The theory treated here is no more than textbook knowledge.¹⁻⁶⁾ Throughout this paper, we take the NLS equation as an illustration:

$$i\psi_t = -\psi_{xx} - 2|\psi|^2\psi. \quad (2)$$

For the NLS equation, the matrices M and N in Eq. (1) are given as

$$M = \begin{bmatrix} -i\xi & i\psi^* \\ i\psi & i\xi \end{bmatrix}, \quad (3a)$$

$$N = \begin{bmatrix} 2i\xi^2 - i|\psi|^2 & \psi_x^* - 2i\xi\psi^* \\ -\psi_x - 2i\xi\psi & -2i\xi^2 + i|\psi|^2 \end{bmatrix}, \quad (3b)$$

where ξ is the spectral parameter. Equations (1a) and (3a) completely define the ZS problem for the NLS equation. Satsuma and Yajima extensively studied this problem and obtained exact results for Asech-type initial conditions, where A is a magnitude of the initial pulse.⁴²⁾ Other soliton equations belonging to the AKNS system have similar ZS problems.

We introduce the usual boundary condition for ψ :

$$\psi \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \quad (4)$$

With this boundary condition, each element of the wave function Ψ must become a plane wave. As the fundamental solutions, we can select two sets of functions $\{\phi, \bar{\phi}\}$ and $\{\chi, \bar{\chi}\}$ called the Jost functions, which satisfy the boundary conditions

$$\phi(x; \xi) \rightarrow \begin{bmatrix} e^{-i\xi x} \\ 0 \end{bmatrix}, \quad \bar{\phi}(x; \xi) \rightarrow \begin{bmatrix} 0 \\ e^{i\xi x} \end{bmatrix}, \quad \text{as } x \rightarrow -\infty, \quad (5a)$$

$$\chi(x; \xi) \rightarrow \begin{bmatrix} 0 \\ e^{i\xi x} \end{bmatrix}, \quad \bar{\chi}(x; \xi) \rightarrow \begin{bmatrix} e^{-i\xi x} \\ 0 \end{bmatrix}, \quad \text{as } x \rightarrow +\infty. \quad (5b)$$

The Jost functions are related to each other as

$$\begin{aligned} \phi(x; \xi) &= a(\xi)\bar{\chi}(x; \xi) + b(\xi)\chi(x; \xi), \\ \bar{\phi}(x; \xi) &= \bar{a}(\xi)\chi(x; \xi) - \bar{b}(\xi)\bar{\chi}(x; \xi). \end{aligned} \quad (6)$$

The coefficient functions are members of the scattering data, and $a(\xi)$ can be analytically continued to the upper half-plane $\text{Im } \xi > 0$.

From Eqs. (5) and (6), we can see that the Jost function $\phi(x; \xi)$ satisfies the asymptotic form

$$\phi(x; \xi) = \begin{bmatrix} a(\xi)e^{-i\xi x} \\ b(\xi)e^{i\xi x} \end{bmatrix}, \quad \text{as } x \rightarrow \infty. \quad (7)$$

When the function $a(\xi)$ has N simple zeros $\xi = \xi_1, \xi_2, \dots, \xi_N$ on the upper half-plane, N solitons appear in the asymptotic future and each ξ determines the characteristics of each soliton. We need to know $a(\xi)$ to extract information on solitons in the asymptotic future. By Eq. (7), we find that this is equivalent to calculating $\phi(x; \xi)$ at $x \rightarrow \infty$ under the boundary condition given by Eq. (5a).

3. Discretization of the Initial Wave Packet and Approximated Scattering Data

In this section, we consider the ZS problem of the NLS equation:

$$\Psi_x = M\Psi, \quad M = \begin{bmatrix} -i\xi & i\psi^* \\ i\psi & i\xi \end{bmatrix}. \quad (8)$$

Since the spectral parameter ξ is a time-independent quantity, we can take ψ in Eq. (8) to be the initial value of the unknown wave packet $\psi(x, 0)$.

The major difficulty in analyzing Eq. (8) for general initial conditions comes from the fact that $\psi(x, 0)$ depends on the coordinate x . In order to overcome this difficulty, according to Boffetta and Osborne's idea,³⁹⁾ we split the support of $\psi(x, 0)$ into many small intervals:

$$I_j : x_j \leq x < x_{j+1} \quad (j = 1, \dots, N), \quad (9)$$

and approximate $\psi(x, 0)$ so that it takes a constant value in each interval. We introduce a set of functions ψ_j :

$$\psi_j(x) = \begin{cases} V_j & x \in I_j, \\ 0 & x \notin I_j. \end{cases} \quad (10)$$

The initial value $\psi(x, 0)$ is now approximated as

$$\psi(x, 0) \simeq \sum_{j=1}^N \psi_j(x), \quad (11)$$

$$= \begin{cases} V_j & (x \in I_j, j = 1, 2, \dots, N), \\ 0 & (\text{otherwise}). \end{cases} \quad (12)$$

Assuming $\psi(x, 0)$ belongs to the class of rapidly decreasing functions, we can approximately consider that $\psi(x, 0)$ has a compact support. Within each interval, Eq. (8) reads as

$$\Psi_x = M_j \Psi, \quad M_j = \begin{bmatrix} -i\xi & iV_j^* \\ iV_j & i\xi \end{bmatrix}. \quad (13)$$

We can solve Eq. (13) for x satisfying $x \in I_j$ as

$$\Psi(x) = T(X)\Psi(x_j), \quad T(X) = \exp(XM_j), \quad (14a)$$

where $X \equiv x - x_j$ and the matrix $T(X)$ is explicitly written as

$$T(X) = \begin{bmatrix} \cos KX - i(\xi/K) \sin KX & i(V_j^*/K) \sin KX \\ i(V_j/K) \sin KX & \cos KX + i(\xi/K) \sin KX \end{bmatrix}, \quad (14b)$$

$$K = \sqrt{\xi^2 + |V_j|^2}. \quad (14c)$$

We denote the width of the j th bin as

$$x_{j+1} - x_j = L_j, \quad (15)$$

and we can see that the Jost function satisfies the relation

$$\Psi(x_{N+1}) = T\Psi(x_1), \quad (16)$$

$$T = T(L_N)T(L_{N-1}) \cdots T(L_2)T(L_1). \quad (17)$$

The matrix T is interpreted as a transfer matrix that connects two asymptotic forms in $x \rightarrow \pm\infty$. Recalling Eq. (5a) and the fact that we truncate $\psi(x; \xi)$ so that it is supported only in the region $x_1 \leq x \leq x_{N+1}$, one can derive the relation

$$\phi(x_{N+1}; \xi) = T\phi(x_1; \xi) = e^{-i\xi x_1} T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (18)$$

$$= \begin{bmatrix} a(\xi)e^{-i\xi x_{N+1}} \\ b(\xi)e^{i\xi x_{N+1}} \end{bmatrix}. \quad (19)$$

Thus, we have the approximated expressions for scattering data in terms of the transfer matrix as

$$a(\xi) = e^{iL\xi} T_{11}, \quad (20a)$$

$$b(\xi) = e^{-i(x_1 + x_{N+1})\xi} T_{21}, \quad (20b)$$

where the parameter L is defined as $L = L_1 + L_2 + \cdots + L_N$. By considering the initial packet as a set of constant functions, one can obtain explicit expressions for the scattering data $a(\xi)$ and $b(\xi)$ for any initial values provided they belong to the class of rapidly decreasing functions. Thus, the desired information that characterizes solitons in the asymptotic future can be extracted from $a(\xi)$ with arbitrary precision by suitably adjusting the width of each bin L_j .

4. Applications to Box-Type Initial Conditions

In this section, we apply Boffetta and Osborne's method introduced in the previous section to box-type initial wave packets. We can solve the ZS problems for these initial conditions exactly, and can derive the corresponding final states explicitly.

Throughout this section, we assume that the initial conditions are real-valued, which means that the initial wave packets are static. Thus, the spectral parameter ξ is expected to be purely imaginary. Since the zeros of $a(\xi)$ should be located in the upper half-plane of ξ , we find the discrete eigenvalues under the condition

$$\xi = i\eta, \quad (\eta > 0). \quad (21)$$

4.1 Single-box-type initial condition

We consider a box-type initial condition whose width is L :

$$\psi(x, 0) = \begin{cases} V_0 & (0 \leq x \leq L), \\ 0 & (\text{otherwise}), \end{cases} \quad (22)$$

where V_0 is a real number. From Eqs. (14b) and (20a), the scattering datum $a(\xi)$ is derived as

$$a(\xi) = e^{i\xi L}(\cos KL - i\frac{\xi}{K} \sin KL), \quad (23)$$

$$K = \sqrt{V_0^2 + \xi^2}.$$

Under the condition (21), we find that the zeros of $a(\xi)$ can be derived from the set of relations

$$\sqrt{V_0^2 - \eta^2} = -\eta \tan(L \sqrt{V_0^2 - \eta^2}), \quad (|V_0| > \eta), \quad (24a)$$

$$\sqrt{\eta^2 - V_0^2} = -\eta \tanh(L \sqrt{\eta^2 - V_0^2}), \quad (|V_0| < \eta). \quad (24b)$$

Since the solution of Eq. (24b) does not satisfy the condition $\eta > 0$, we eliminate it and consider only Eq. (24a). Introducing A and u as

$$A = V_0 L, \quad u = \eta L, \quad (25)$$

we can omit the parameter L . Thus, the equation we should consider becomes

$$\sqrt{A^2 - u^2} = -u \tan(\sqrt{A^2 - u^2}), \quad (26a)$$

$$0 < u < A. \quad (26b)$$

This means that we should find the intersection of the curves $y = -u \tan \sqrt{A^2 - u^2}$ and $y = \sqrt{A^2 - u^2}$ in the first quadrant.

For a sufficiently minute value of V_0 , the value of $\sqrt{A^2 - u^2}$ remains in the interval $(0, \pi/2)$ and the right-hand side of (26a) remains negative. In such a case, no solutions exist and no solitons remain at $t \rightarrow \infty$. As the value of V_0 increases, $\sqrt{A^2 - u^2}$ can exceed $\pi/2$ and solutions of Eq. (26a) appear. It is clear that the condition under which Eq. (26) has at least one solution is $A > \pi/2$ ($V_0 > \pi/(2L)$). Typical situations in which these cases occur are shown in Fig. 1. After a brief consideration, we can see that if the potential height V_0 satisfies

$$(n - \frac{1}{2})\frac{\pi}{L} < V_0 \leq (n + \frac{1}{2})\frac{\pi}{L}, \quad (n: \text{a positive integer}), \quad (27)$$

the number of solitons that will remain over time should be n .

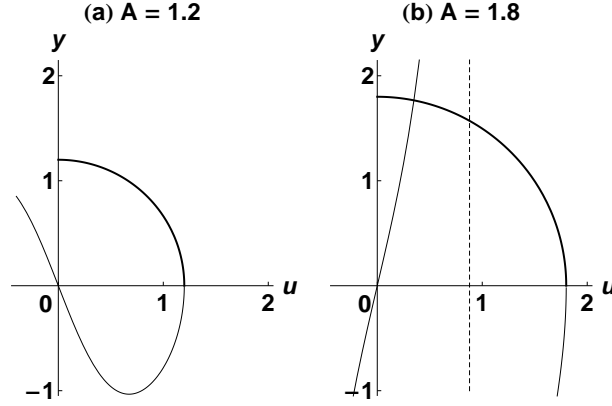


Fig. 1. Typical cases concerning Eq. (26). (a) Case of $A = 1.2$. (b) Case of $A = 1.8$. The thick curves are the graphs of $y = \sqrt{A^2 - u^2}$ in the region $u > 0$, and the thin curves are the graphs of $y = -u \tan \sqrt{A^2 - u^2}$. The dashed line in (b) denotes the value of u where $\sqrt{A^2 - u^2} = \pi/2$ and the right-hand side of Eq. (26a) diverges.

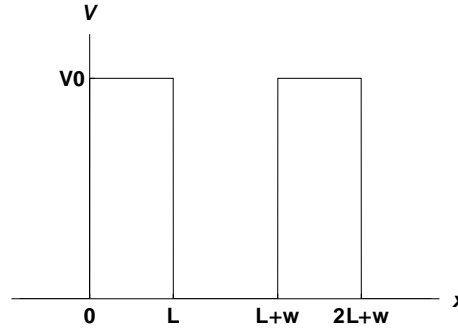


Fig. 2. Double-box-type initial condition (28).

4.2 Double-box-type initial condition

Next, we consider the initial condition

$$\psi(x, 0) = \begin{cases} V_0, & (0 < x < L, L + w < x < 2L + w), \\ 0, & (\text{otherwise}), \end{cases} \quad (28)$$

which means that two identical pulses, each of which has a common amplitude V_0 and width L , are located with separation w (Fig. 2). In this case, by using Eq. (21), we find that the scattering amplitude $a(\xi)$ is given by

$$a(i\eta) = e^{-2\eta L} \left\{ \left[\cos KL + \frac{\eta}{K} \sin KL \right]^2 - \frac{V_0^2}{K^2} e^{-2\eta w} \sin^2 KL \right\}, \quad (29a)$$

$$K \equiv \sqrt{V_0^2 - \eta^2}. \quad (29b)$$

If we introduce A and u as in Eq. (25), the zeros of $a(i\eta)$ can be derived from

$$\left[\cos K + \frac{u}{K} \sin K \right]^2 = \frac{A^2}{K^2} e^{-2uw/L} \sin^2 K, \quad (30a)$$

$$K = \sqrt{A^2 - u^2}. \quad (30b)$$

The values of u are restricted to

$$0 < u < A, \quad (30c)$$

because there are no positive η satisfying this relation if $\eta > V_0$, following the discussion in deriving Eq. (26b).

Let us consider two limiting cases. When two initial pulses are sufficiently separated, it is natural to expect that the number of solitons that remain over time will be twice that in the case of a single initial pulse, because the amplitude of diffusing radiation is generally so small that the interaction between the two pulses hardly affects the asymptotic future. This observation is confirmed by taking the limit $w \rightarrow \infty$ in Eq. (29a). This operation makes the final term in Eq. (29a) vanish. In this limit, the function a given by Eq. (29a) coincides with the square of the scattering amplitude of Eq. (23) under the condition (21). In the opposite limit, $w \rightarrow 0$, the two initial pulses are fused together into a single pulse whose width is $2L$. In fact, Eq. (29a) coincides with Eq. (23) if L is replaced by $2L$.

For an appropriately chosen value of w , the analysis of the eigenvalue problem provides nontrivial solutions where the final term of Eq. (29a) plays an essential role. We have shown in Fig. 3 the curves that satisfy Eq. (30a) on the A - u plane. We have chosen the values of separation as $w = 0.1L$ in Fig. 3(a) and $w = 1.5L$ in Fig. 3(b). In Fig. 3(a), we can see that there is no solution for $A \lesssim 0.8$. This means that an initial wave packet with a very small amplitude is completely transformed into diffusing waves known as radiation. As the value of A ($\sim V_0$) increases, a solution of Eq. (30a) appears. This occurs when $\pi/4 \lesssim A \lesssim 3\pi/4$. This solution gives one asymptotically remaining soliton. The soliton is expected to be located midway between the two initial pulses because of the spatial reflection symmetry. For larger values of A , the number of remaining solitons increases monotonically.

In Fig. 3(b), we can see that there is a qualitatively different result that cannot be observed in the previous case. In this case, the quantity η is not always given as a single-valued function of A on every branch. After having one solution for $\pi/4 \lesssim A \lesssim 2.2$, Eq. (30a) is observed to have two roots around $A \sim 2.2$. The smaller solution of u for this value of A gives a

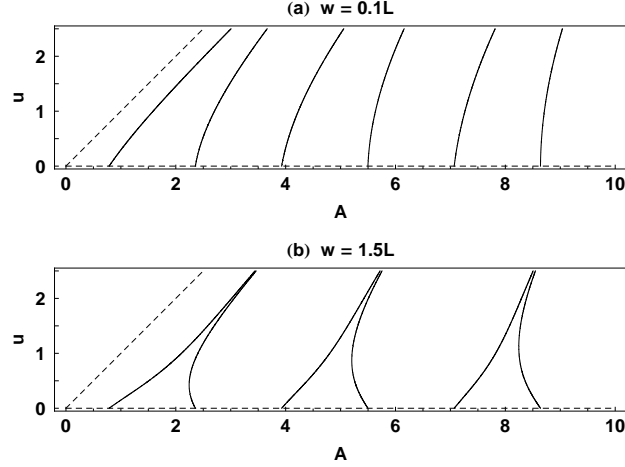


Fig. 3. Distribution of the zeros of scattering datum $a(i\eta)$ for various values of A under the given w . (a) Case of $w = 0.1L$. (b) Case of $w = 1.5L$. The dashed line expresses the two boundaries of the allowed region for the solutions, given by Eq. (30c).

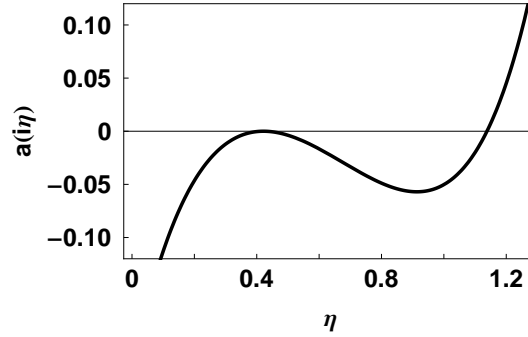


Fig. 4. Graph of $a(iu)$ for $A = A_0$ (2.24228). A double-pole solution of $a(iu) = 0$ appears.

double-pole solution. We have presented a graph of $a(i\eta)$ in Fig. 4 for the value of $A \simeq 2.2$, the smallest value of A where the tangent of the curves shown in Fig. 3(b) is parallel to the u -axis. Let us denote this value of A as A_0 . When $A = A_0$, the number of remaining solitons is two. If A exceeds this value, the number of solitons first becomes three. As the value of A becomes larger, the number of remaining solitons decreases to two for $A \gtrsim 3\pi/4$. Thus, the number of solitons that appears over time is not a simple monotonic function of the amplitude of the initial pulse for a moderate value of w .

4.3 Asymmetric double-box-type initial condition

In a real experiment on a self-focusing BEC system,²⁹⁾ Nguyen *et al.* prepared two condensates so that the population of one condensate was about half that of the other condensate and made them collide with each other. To give the two condensates opposite initial velocities,

a harmonic potential was applied in the axial direction and they did not turn off the harmonic potential throughout the runs. Unfortunately, the harmonic potential was not so weak that its effect could be neglected because the two condensates were observed to pass through each other and oscillate back and forth at the bottom of the trap for several periods.

Nevertheless, from the standpoint of nonlinear wave theory, it is very interesting to consider this problem under the ideal potential-free situation. Although the exact initial conditions include smooth shapes and phases of the condensates, we modeled the two condensates as an asymmetric double box-type initial condition on the flat line for simplicity as below:

$$\psi(x, 0) = \begin{cases} V_0, & (0 < x < L), \\ sV_0, & (L + w < x < 2L + w), \\ 0, & (\text{otherwise}). \end{cases} \quad (31)$$

Assuming $\xi = i\eta$ ($\eta > 0$) and using the normalized parameters $A = LV_0$ and $u = L\eta$ as before, we define the following functions:

$$f(A, u) = \cos K(A) + \frac{u}{K(A)} \sin K(A), \quad (32a)$$

$$g(A, u) = \frac{A}{K(A)} \sin K(A), \quad (32b)$$

$$K(A) = \sqrt{A^2 - u^2}. \quad (32c)$$

The scattering amplitude $a(i\eta)$ for the initial condition (31) is now expressed as

$$a(iu) = e^{-2u} \left(f(A, u)f(sA, u) - e^{-2\frac{uw}{L}} g(A, u)g(sA, u) \right), \quad (33)$$

which is a symmetric function of V_0 and sV_0 as expected, and the last term exactly describes the effect of interacting tails. This time, we should note that $f(sA, u)$ and $g(sA, u)$ include hyperbolic functions when $sA < u$. In Fig. 5, we show the curves $a(iu) = 0$ on the A - u plane obtained by setting $w = 1.5L$ as before and assuming $s = 1/\sqrt{2}$. This selection of s corresponds to the situation where the population of the condensate with a small amplitude is half that of the other condensate, as in the experiment described in Ref. 29. As A ($\sim V_0$) increases, we can see, in addition to the fact that η is not always given as a single-valued function of A on every branch, that some of the branches intersect each other. The first intersection appears at around $A = 6.63481$. This point gives a double-pole soliton condition, although the measure of this point is no more than zero. We have presented a graph of $a(iu)$ in Fig. 6 for this value of A .

The appearance of these intersections clearly makes the trajectory $a(iu) = 0$ more com-

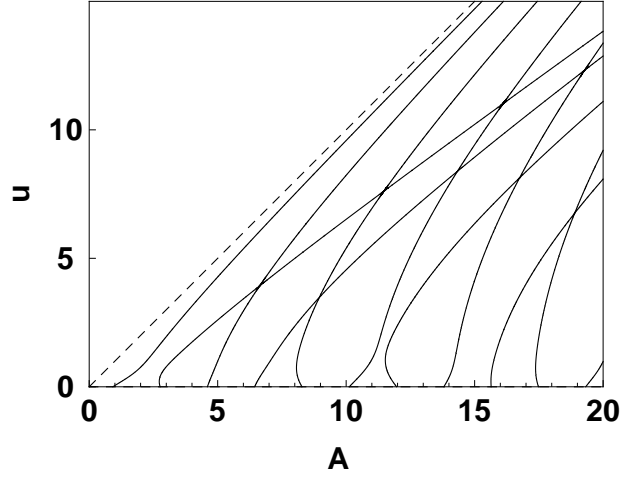


Fig. 5. Distribution of the zeros of scattering amplitude $a(iu)$ for various values of A under $w = 1.5L$ and $s = 1/\sqrt{2}$. The dashed line expresses the two boundaries of the allowed region for the solutions, given by Eq. (30c).

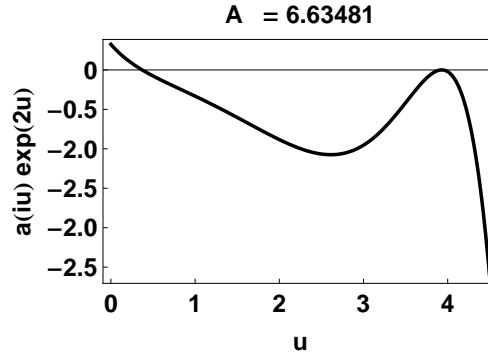


Fig. 6. Graph of $a(iu)\exp(2u)$ for $A = 6.63481$. A double-pole solution also appears at the crossing point.

plex and richer than that in the symmetric case. The number of solitons that appears over time is again not a simple monotonic function of the amplitude of the initial pulse for a suitably chosen value of w .

5. Numerical Simulation

We show the results of the numerical simulation of the initial value problem analyzed in the previous subsection. By numerically integrating the NLSE (2), we solve the initial value problem under the double-box-type initial condition (28) horizontally shifted so that the center of the valley coincides with the origin $x = 0$. We set the width of the valley w to $1.5L$ and vary the common potential height V_0 for each time.

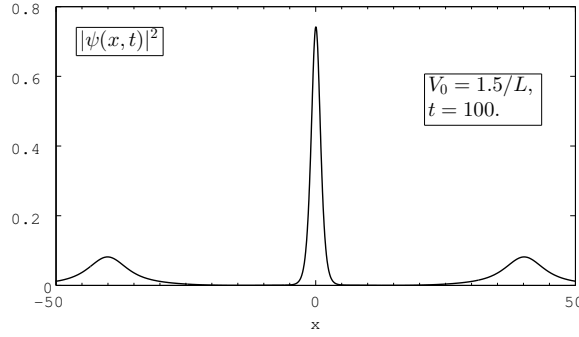


Fig. 7. Graph of $|\psi(x, t)|^2$ at $t = 100$ for $V_0 = 1.5/L$.

First, we examine the case where $V_0 = 1.5/L$. In this case, we have only one solution so that we expect only one soliton in the final state. Figure 7 shows the absolute square of the wave amplitude, $|\psi(x, t)|^2$, at $t = 100$. Although we can observe three pulses, the height of the two smaller peaks relative to that of the center peak keeps decreasing and fades away. Only one soliton at the center is expected to remain at the limit $t \rightarrow \infty$.

Secondly, we increase the height of the potential to $V_0 = 2.3/L$, which is slightly larger than the critical value for a double-pole soliton but smaller than the upper threshold $V_0 = 3\pi/4$. Therefore, we expect three remaining solitons in the far distant future. Figure 8 shows the value of $|\psi(x, t)|^2$ at $t = 65$. We can observe three sharp pulses. In this case, the two smaller peaks at both sides do not vanish.

Thirdly, we set the potential height to $V_0 = 2.5/L$, which exceeds the boundary value of $V_0 = 3\pi/4$, and two solitons are predicted to survive. Figure 9 shows the absolute square of the wave amplitude $|\psi(x, t)|^2$ at $t = 50$. We can observe two large peaks around the origin as expected. These peaks keep alternately splitting and fusing together, like a breather solution.

In the case of a symmetric initial condition, we also observed similar behaviors. Unfortunately, our numerical simulation could not capture the features of the double-pole solitons. This is reasonable because the conditions for generating them have zero measures. Other results of the numerical simulation, however, show good agreement with theoretical predictions and strongly support their validity.

6. Discussion and Concluding Remarks

Inspired by a recent collision experiment using BEC pulses, we have applied Boffetta and Osborne's approximation method to analyze the Zakharov-Shabat eigenvalue problem,

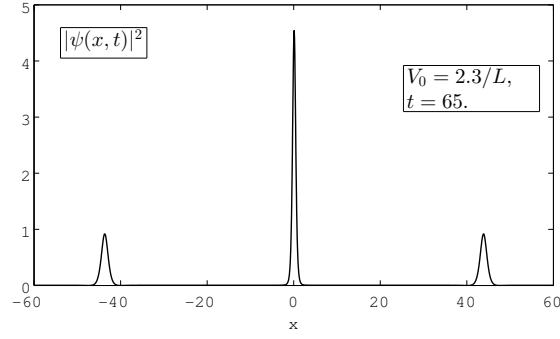


Fig. 8. Graph of $|\psi(x, t)|^2$ at $t = 65$ for $V_0 = 2.3/L$.

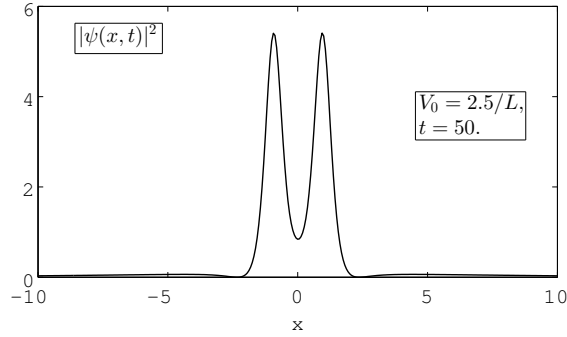


Fig. 9. Graph of $|\psi(x, t)|^2$ at $t = 50$ for $V_0 = 2.5/L$.

which is associated with the NLSE. As we have seen in this paper, their method can be used effectively in solving the ZS eigenvalue equations under general nonsoliton initial conditions. We have considered the initial value problem of the self-focusing-type NLS equation under box-type initial conditions, for which we can analytically obtain exact results. We have found that the interplay between the decaying tails from the initial pulses can affect the asymptotic behaviors, and we succeeded in making qualitative predictions including the number of remaining solitons and conditions under which the initial wave becomes double-pole solitons. In particular, under an asymmetric initial condition where the boxes have different heights, further complex and interesting behaviors have been observed.

Needless to say, the double-box-type initial conditions that we used are idealized mathematical models and we omitted the external trap potential in the axial direction. Our results, however, are not so far from actual experimental conditions. In the first place, it seems possible to turn off or loosen the axial potential immediately after creating two BEC pulses, so that they can freely interfere with each other. In fact, Nguyen *et al.* confirmed that their bright BEC

soliton did not diffuse and stayed still in an almost flat axial geometry.²⁹⁾ The experimental realization of a BEC bright soliton means that, from Eq. (27), the key parameter $A = LV_0$ satisfies $\pi/2 < A < 3\pi/2$. We have shown that double pulses with moderate separation develop into a three-soliton state if each pulse satisfies $A \sim 2.25$. As we have already mentioned, at least $A \geq \pi/2 = 1.57$ is already feasible. To achieve $A \sim 2.25$ without altering the pulse width L , we should have an approximately 1.43-fold larger value of V_0 , which is roughly equivalent to double the total number of atoms. To avoid instability or self-collapse of attractively interacting BEC systems, one must keep the total atom number less than the critical value of $N_c = 0.67a_r/|a_s|$,⁴³⁾ where a_r and a_s are the radial trap size and the s-wave scattering length, respectively. If we can double the radial trap size a_r , we can satisfy $A \sim 2.25$. According to Ref. 29, we can calculate the value of a_r and obtain $a_r \sim 6\mu\text{m}$. In the observation of the first bright soliton trains of the same ^7Li atoms, their radial trap size was reported to be about $47\mu\text{m}$.²⁴⁾ Hence, $a_r = 12\mu\text{m}$ seems to be an experimentally sound parameter.

Finally, we refer to the possible extensions of this work. Although we have limited ourselves to the consideration of the NLS equation, this method can be applied to various soliton equations that belong to the AKNS system. In addition, we can expect more extensions to integrable equations that belong to other systems, such as the Kaup-Newell⁴⁴⁾ and Wadati-Konno-Ichikawa⁴⁵⁾ systems. These extensions should be considered as future works, and more interesting physics brought about by nonlinear wave interaction is expected to be extracted in an analytically accessible manner.

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